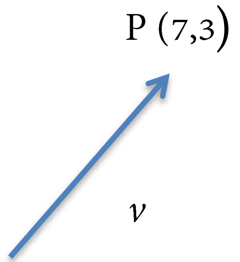
Computer Graphics

Lecture 21

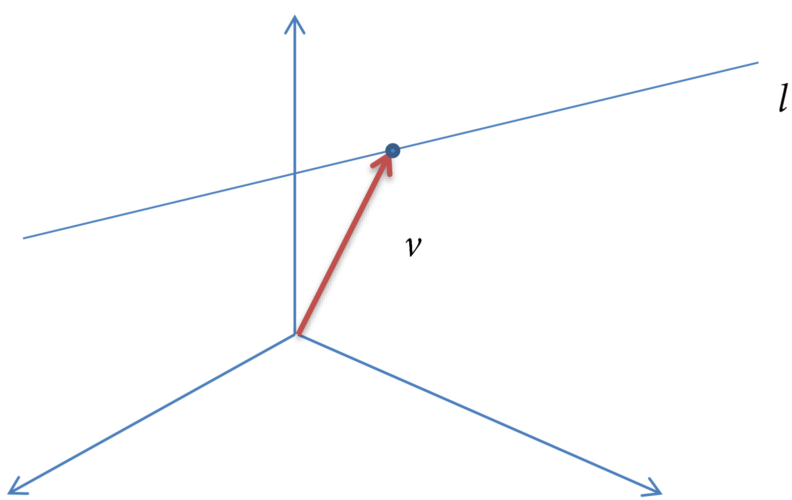
Parametric equation of a Line

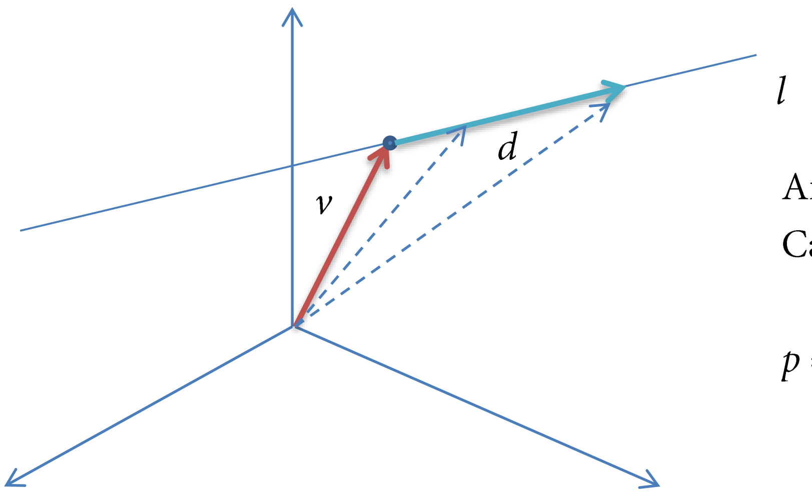


v = <7,3>

v is called a position vector of the point (7,3)

Find a equation of a line in terms of position vector that lies on the line





Any given position vector *p*
Can be represented by

$p = v + td$

where t is a scalar

Example: Find the equation of line passing (1,-1,4) in direction of <3,5,-1>

p = <1,-1,4> + t<3,5,-1>

x = 1+3t

y = -1+5t

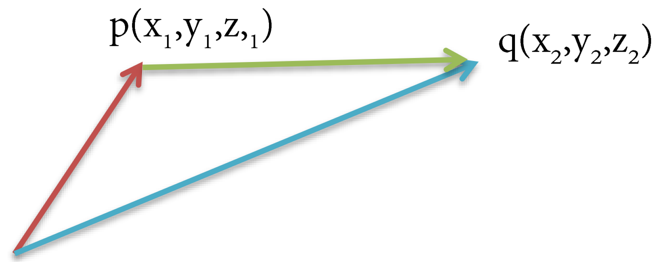
z = 4-t

Likewise line between $p(x_1,y_1,z,_1)$ and $q(x_2,y_2,z_2)$ given by

$x = x1 + t(x_2 - x_1)$

$y = y1 + t(y_2 - y_1)$

$z = z1 + t(z_2 - z_1)$



Can denote as

$x = x_1 + t*dx$

$y = y_1 + t*dy$

$z = z_1 + t*dz$

Find intersect point of line and a plane

Given plain $x+2y+z = 6$;

Line move from (1,2,3) and (4,3,8)

Steps:

- Find x,y,z in terms of t
- Place x,y,z plain equation and solve for t
- find appropriate points by applying t to x,y,z

Barycentric Coordinates

Triangles are the fundamental primitive used in 3D modeling programs.

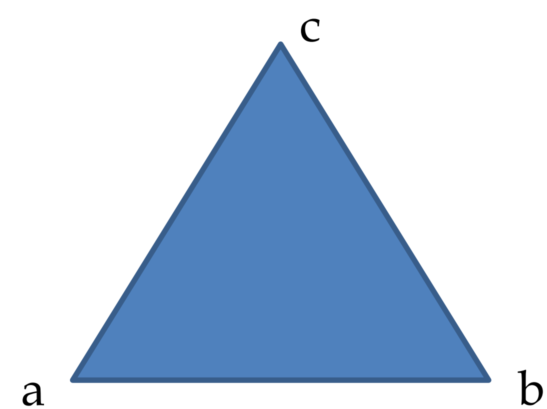
We can express any point p coplanar to the Triangle as

$P = a + \beta(b-a) + \lambda(c-a)$

We rewrite this as

$P(\alpha,\beta,\lambda) = \alpha a + \beta b + \lambda c$ *where*

$\alpha \equiv 1 - \beta - \lambda$



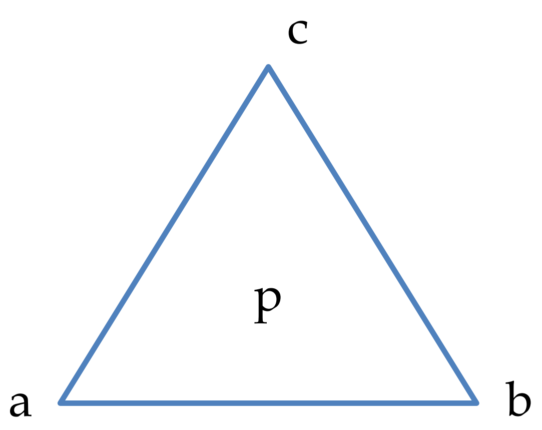
Example

Let

a = p(1,0,0)

b = p(0,1,0)

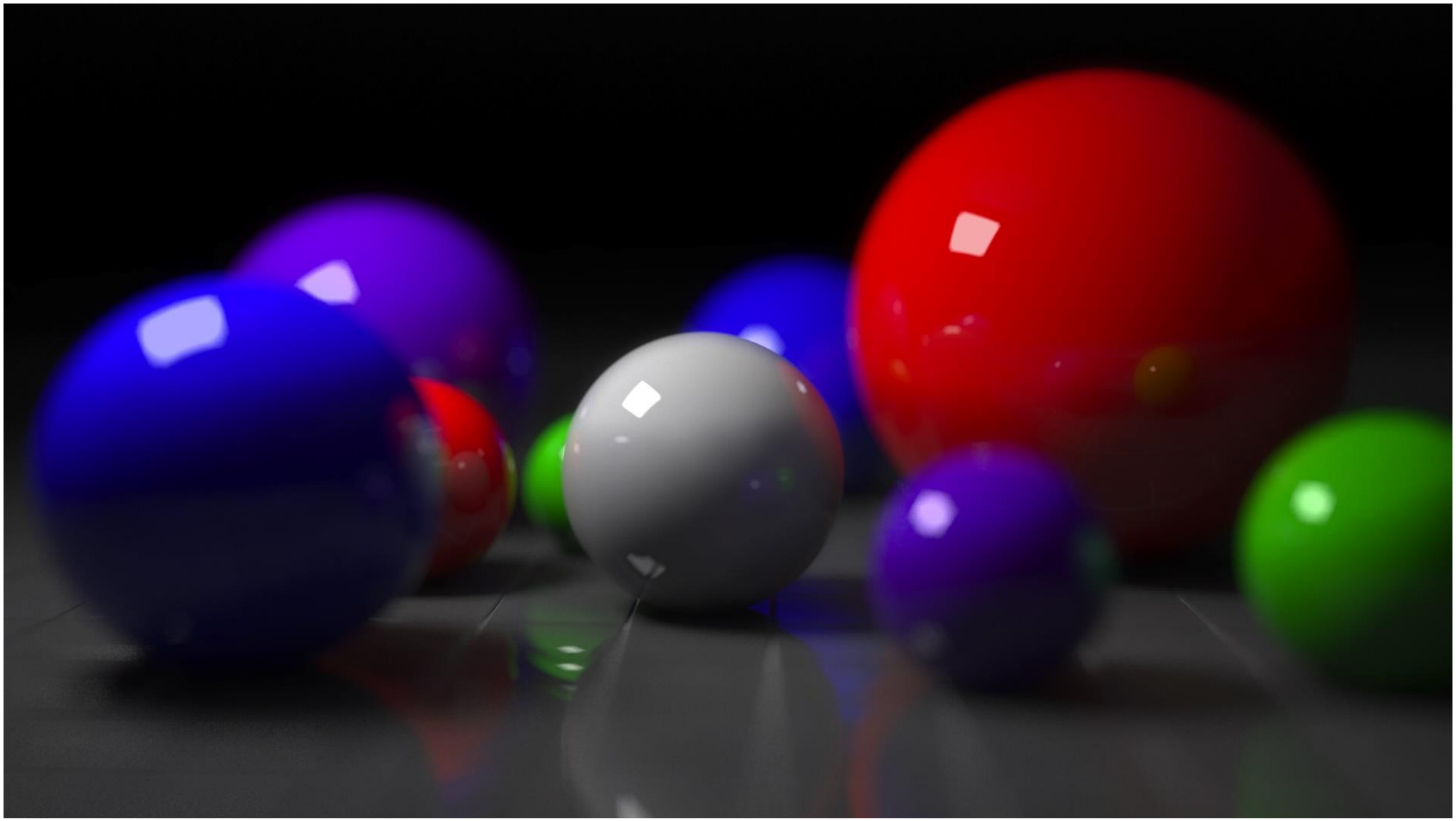
c = p(0,0,1)



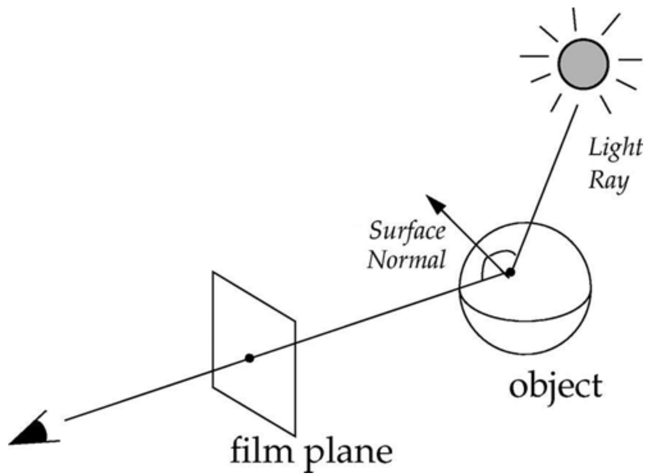
Point p is inside the triangle if
$0 < \alpha < 1$
$0 < \beta < 1$
$0 < \lambda < 1$

- If one component is zero
  - P is on an edge
- If two components are zero
  - P is on a vertex

Ray Tracing



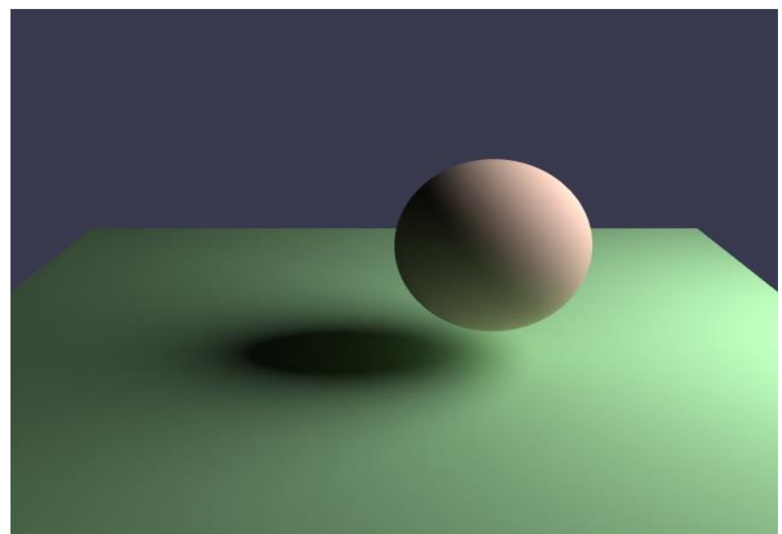
Ray Tracing Fundamentals



- Generate primary ray
  - shoot rays from eye through sample points on film plane
  - sample point is typically center of a pixel
- Ray-object intersection
  - find first object in scene that ray intersects with (if any)
  - use parametric line equation for ray, so smallest t value
- Calculate lighting (i.e., color)
  - use illumination model to determine direct contribution from light sources (light rays)
  - reflective objects recursively generate secondary rays
  - No diffuse reflection rays => RT is limited approximation to global illumination

Recursive ray tracing

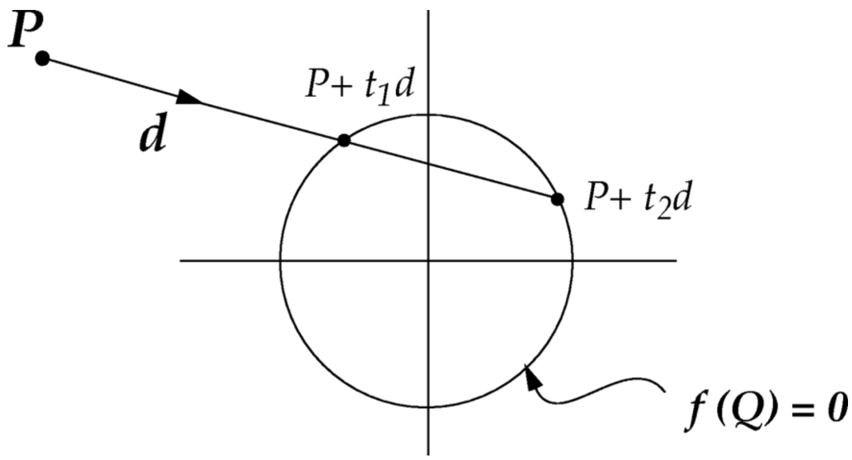
http://learningthreejs.com/blog/2012/01/20/casting-shadows/



Ray-Object Intersection

At what points (if any) does the ray intersect an object?

- Points on a ray have form $\boldsymbol{P} + t\boldsymbol{d}$ where t is any non-negative real number
- A surface point Q lying on an object has the property of *f(Q)*
- Combining, we want to know "For which values of t is $f(\boldsymbol{P} + t\boldsymbol{d}) =$?"



- We are solving a system of simultaneous equations in x, y (in 2D) or x, y, z (in 3D)

2D ray-circle intersection example

- Consider the eye-point $\boldsymbol{P}$ = (-3, 1), the direction vector $\boldsymbol{d}$ = (.8, -.6) and the unit circle: $f(x,y) = x^2 + y^2 - R^2$
- A typical point of the ray $Q = \boldsymbol{P} + t\boldsymbol{d} = (-3,1) + t(.8,-.6) = (-3 + .8t, 1 - .6t)$
- Plugging this into the equation of the circle:
  $f(Q) = f(-3 + .8t, 1 - .6t) = (-3+.8t)^2 + (1-.6t)^2 - 1$
- Expanding, we get: $9 - 4.8t + .64t^2 + 1 - 1.2t + .36t^2 - 1$
- Setting this equal to zero, we get: $t^2 - 6t + 9 = 0$

- Using the quadratic formula: $roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- We get: $t = \frac{6 \pm \sqrt{36 - 36}}{2}, \qquad t = 3, 3$
- Because we have a root of multiplicity 2, ray intersects circle at only one point (i.e., it's tangent to the circle)
- Use discriminant D = $b^2 - 4ac$ to quickly determine if true intersection:
  - if $D < 0$, imaginary roots; no intersection
  - if $D = 0$, double root; ray is tangent
  - if $D > 0$, two real roots; ray intersects circle at two points

Barycentric ray-triangle intersection

Every point on the plane can be written in the form

$$\mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

for some numbers β and γ.

If the point is also on the ray then it is

$$\mathbf{p} + t\mathbf{d}$$

for some number t.

Set them equal: 3 linear equations in 3 variables

$$\mathbf{p} + t\mathbf{d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

Barycentric ray-triangle intersection

$$\mathbf{p} + t\mathbf{d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

$$\beta(\mathbf{a} - \mathbf{b}) + \gamma(\mathbf{a} - \mathbf{c}) + t\mathbf{d} = \mathbf{a} - \mathbf{p}$$

$$\begin{bmatrix} \mathbf{a} - \mathbf{b} & \mathbf{a} - \mathbf{c} & \mathbf{d} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} \mathbf{a} - \mathbf{p} \end{bmatrix}$$

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_d \\ y_a - y_b & y_a - y_c & y_d \\ z_a - z_b & z_a - z_c & z_d \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x_a - x_p \\ y_a - y_p \\ z_a - z_p \end{bmatrix}$$

Cramer's Rule

$2x + y + z = 3\ ,\quad x - y - z = 0\ ,\quad x + 2y + z = 0$ **find x,y,z**

Evaluating each determinant of the following

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (-2)+(-1)+(2) -(-1)-(-4)-(1) = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (-3)+(0)+(0) -(0)-(-6)-(0) = -3+6 = 3$$

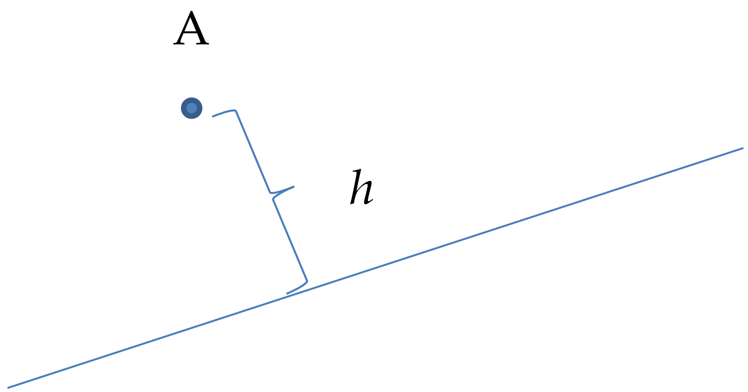
$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = (0)+(-3)+(0) -(0)-(0)-(3) = -3-3 = -6$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (0)+(0)+(6) -(-3)-(0)-(0) = 6+3 = 9$$

x = Dx/D,
y = Dy /D,
z = Dz / D

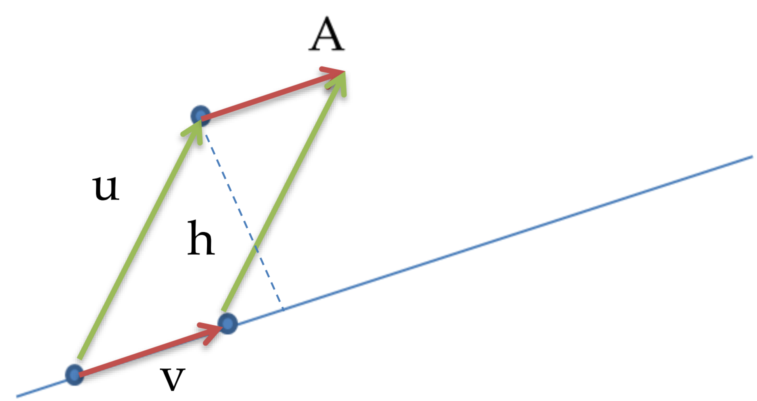
x = 3/3 = 1,
y = -6/3 = -2,
z = 9/3 = 3

Distance between point to a line



Area of parallelogram = || u x v || = base.h
= ||v||.h

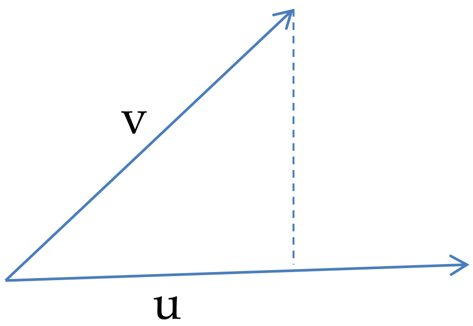
h = || u x v|| / ||v||



Example

Distance from (1,4,-2) to line containing

(3,1,-2) & (6,-2,1)



$v = \langle 1-3,\ 4-1,\ -2+2 \rangle$

$= \langle -2,3,0 \rangle$

$u = \langle 6-3,\ -2-1,\ 1+2 \rangle$

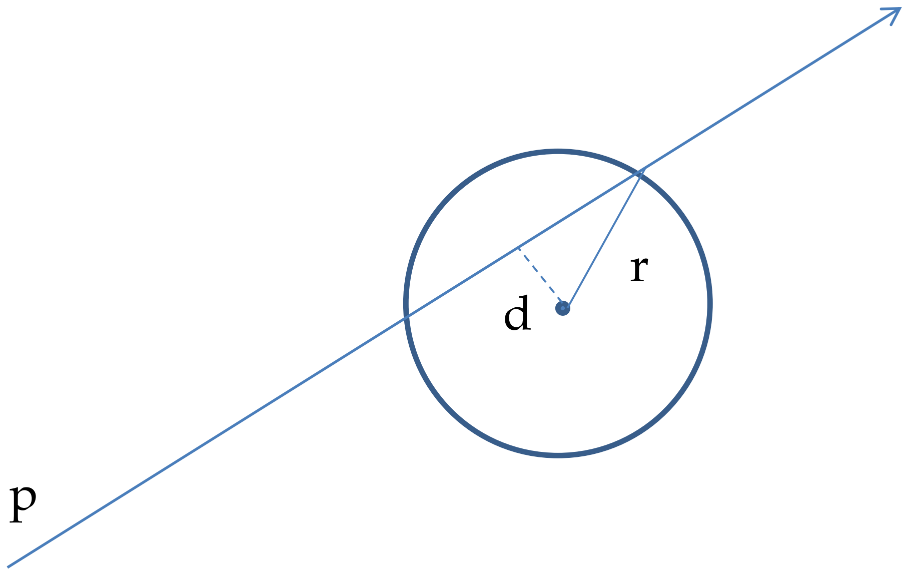
$= \langle 3,\ -3,3 \rangle$

$v \times u = \langle 9,6,-3 \rangle$

$\text{Dist} = ||v \times u|| / ||u||$

$= \sqrt{14/3}$

Ray Intersect Sphere



If d < r ray hits the sphere