Computer Graphics

Lecture 14

What is Bézier curve

A Bézier curve is a parametric curve that is used to draw smooth lines

Named after Pierre Bézier who used them for designing cars at Renault, actually invented by Paul de Casteljau 3 years earlier whilst working for Citroën

Common applications include CAD software, 3D modelling and typefaces

An n degree Bézier curve is defined using $n + 1$ **control points**

Translations can be easily applied to the control points

Bézier curve

- Parametric Equation of the line

Line (x_1, y_1) *to* (x_2, y_2)

Consider t: range from 0 to 1

At Start of the line (x_1, y_1) t=0

At End of the line (x_2, y_2) t=1

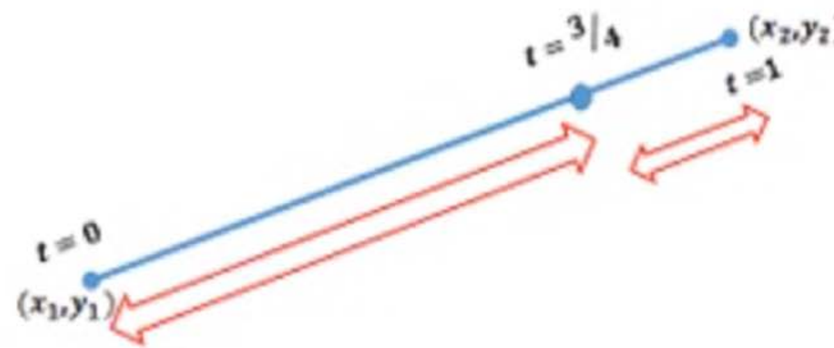
At ${}^3/_4{}^{th}$ of the line path

$t={}^3/_4$

The location is

$x={}^1/_4\, x_1 + {}^3/_4\, x_2$

$y={}^1/_4\, y_1 + {}^3/_4\, y_2$



At time t

x=(1-t). $x_1 + t.x_2$

y=(1-t). y_1 +$t.y_2$

x=$x_1 - x_1.t + t.x_2$

=x_1 + t($x_2 - x_1$)

$$x = x_1 + t\Delta x$$

y=$y_1 - y_1.t + t.y_2$

=y_1 + t($y_2 - y_1$)

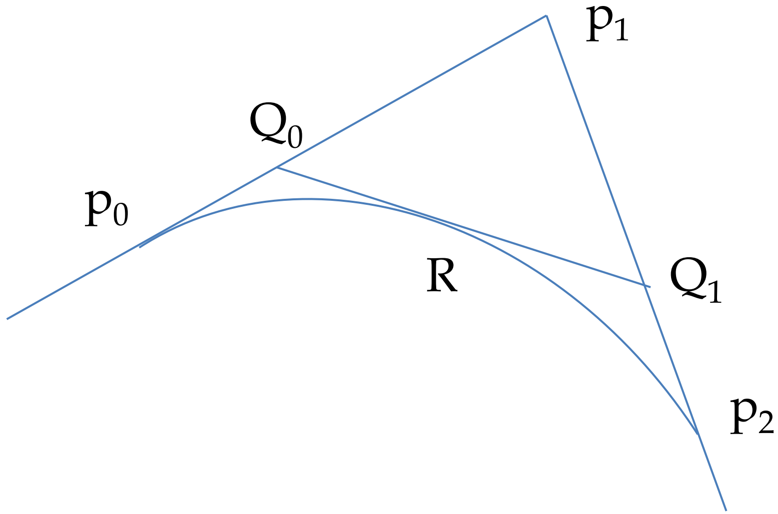
$$y = y_1 + t\Delta y$$

Parametric Line Equation is

$$x = x_1 + t\Delta x$$
$$y = y_1 + t\Delta y$$

Idea behind Bézier curve

Let p_0, p_1 and p_1,p_2 be two convex combinations

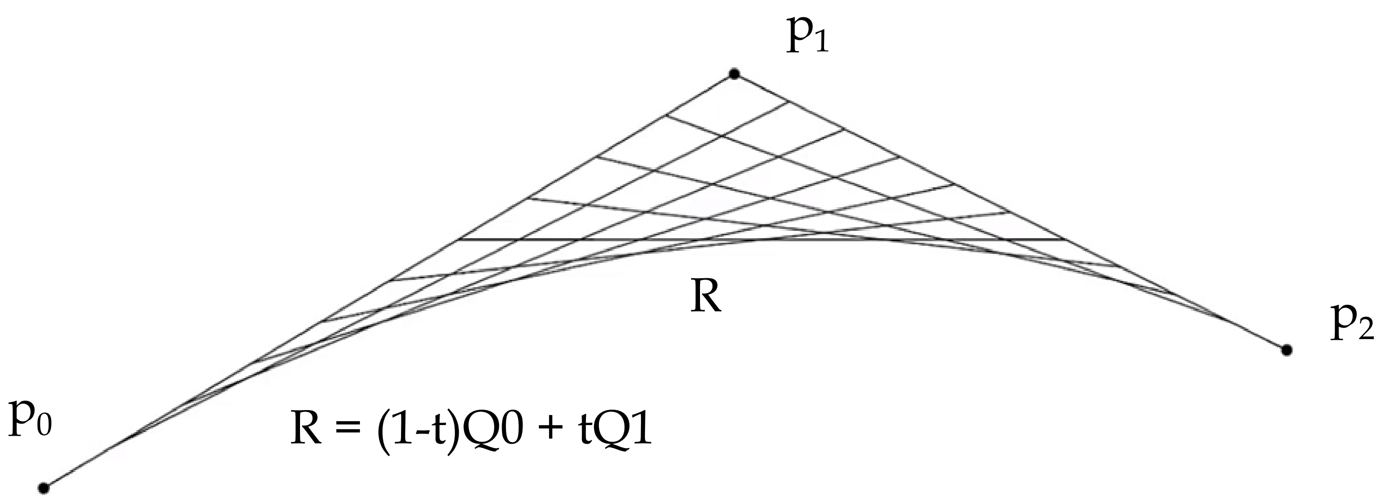


$Q_0 = (1\text{-}t)p_0+tp_1$

$Q_1 = (1\text{-}t)p_1+tp_2$

$R = (1\text{-}t)Q_0 + tQ_1$

As t goes from 0 to 1, R moves along a parabolic path from p_0 to p_2.



R = (1-t)Q0 + tQ1

Mapping R when t moves from 0 to 1

$R = (1-t)[(1-t)p_0+tp_1]+ t[(1-t)p_1+tp_2]$

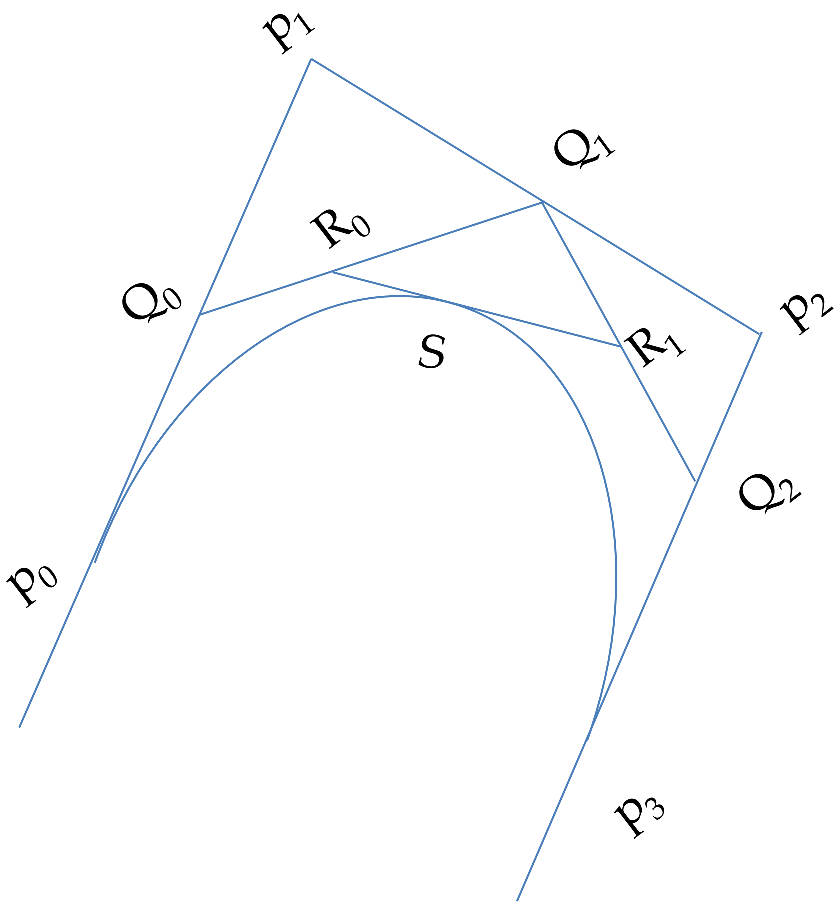
$= (1-t)^2p_0+ 2(1-t)tp_1+t^2p_2$

Since $(1-t)^2 + 2(1-t)t+t^2 =1$ (Binomial theorem)

R consider as convex combination of p_0, p_1, p_2 where

$0 \leq t \geq 1$ and $(1-t)^2 \geq 0$, $2(1-t)t \geq 0$, $t^2 \geq 0$

Cubic Bézier curve



p_0, p_1, p_2, p_3 are fixed points

$Q_0 = (1\text{-}t)p_0 + tp_1$

$Q_1 = (1\text{-}t)p_1 + tp_2$

$Q_2 = (1\text{-}t)p_2 + tp_3$

$R_0 = (1\text{-}t)Q_0 + tQ_1$

$R_1 = (1\text{-}t)Q_0 + tQ_2$

$S = (1\text{-}t)R_0 + tR_1$

When t moves from 0 to 1, S trace out the fine Curv

$$R_0 = (1-t)^2 p_0 + 2(1-t)tp_1 + t^2 p_2$$

$$R_1 = (1-t)^2 p_1 + 2(1-t)tp_2 + t^2 p_3$$

$$S = (1-t)R_0 + tR_1$$

$$S = (1-t)[(1-t)^2 p_0 + 2(1-t)tp_1 + t^2 p_2] + t[(1-t)^2 p_1 + 2(1-t)tp_2 + t^2 p_3]$$

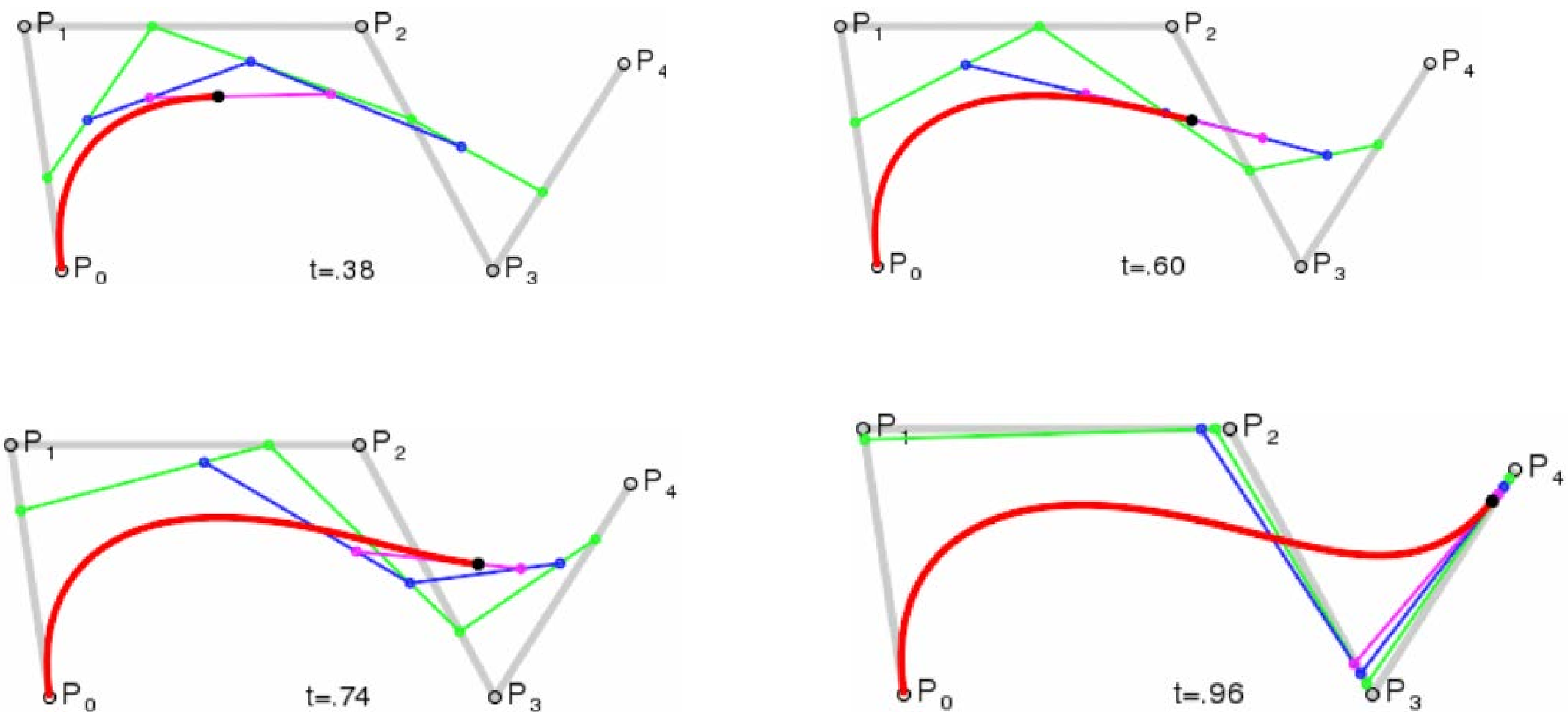
$$S = (1-t)^3 p_0 + 3(1-t)^2 tp_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

Since $(1-t)^3 p_0 + 3(1-t)^2 tp_1 + 3(1-t)t^2 p_2 + t^3 p_3 = 1$ (Binomial theorem)

S consider as convex combination of $p_{0,}\ p_{1,}\ p_{2,}\ p_3$
where

$$0 \leq t \geq 1 \text{ and } (1-t)^3 \geq 0,\ 2(1-t)^3 t \geq 0,\ t^3 \geq 0$$

Higher degree of Bézier curve



More control points leads into higher degree Bézier curve
→ Higher degree Bernstein Polynomials

polynomials

A polynomial of degree n is a function of the form

$$f(x) = a_n x^{n} + a_{n-1} x^{n-1} + \ldots + a_2 x^{2} + a_1 x + a^0$$

In other words

$$\sum_{i=0}^{n} a_i x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Polynomials of different degrees

$f(x) = x + 1,$	linear
$g(x) = x^2 + x + 1,$	quadratic
$h(x) = x^3 + x^2 + x + 1,$	cubic
$i(x) = x^4 + x^3 + x^2 + x + 1.$	quartic

De Casteljau Algorithm

The derivation process use here is known as **de Casteljau's algorithm**.

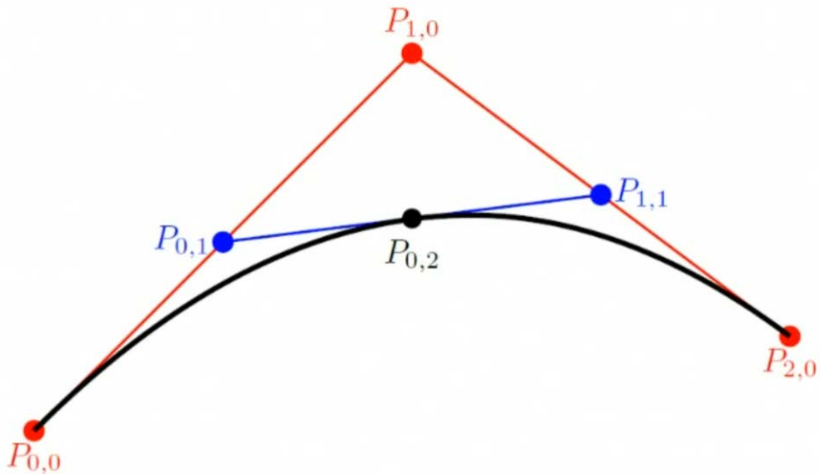
Let $P_{i,j}$ denote the control points where $P_{i,0}$ are the original control points P_0 to P_2, $P_{i,1}$ are the points Q_0 to Q_1 and $P_{0,2}$ is $C(t)$ then

$$P_{i,j} = (1-t)P_{i,j-1} + tP_{i+1,j-1}.$$

$P_{i,j}$ depends on the points $P_{i,j-1}$ and $P_{i+1,j-1}$, i.e.,

$$\begin{array}{ccc} P_{i,j-1} & \rightarrow & P_{i,j} \\ & \nearrow & \\ P_{i+1,j-1} & & \end{array}$$

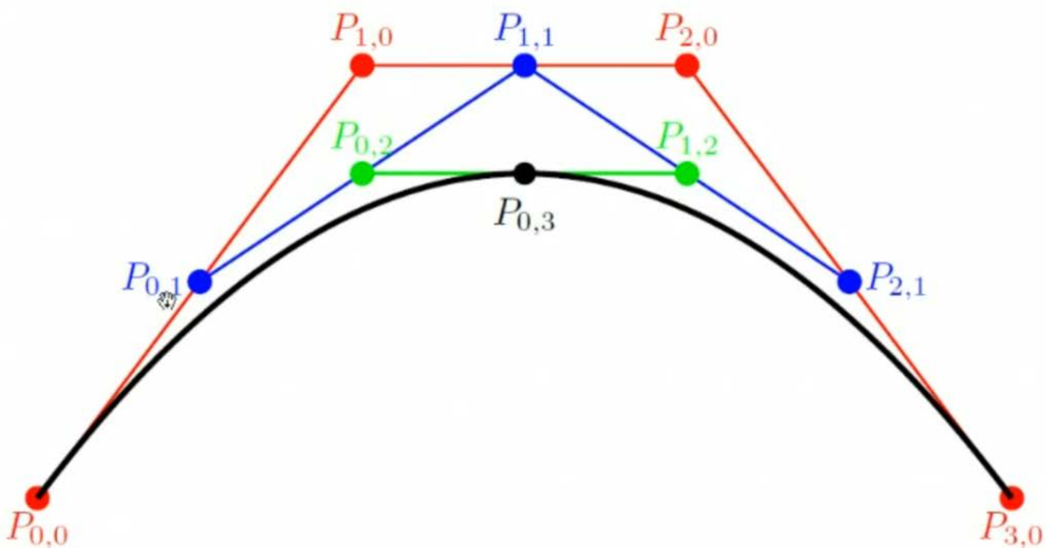
where the horizontal arrow denotes the $(1-t)$ coefficient and the diagonal arrow denotes the t coefficient



$$P_{i,j} = (1-t)P_{i,j-1} + tP_{i+1,j-1}$$

A cubic Bézier curve is defined by 4 control points: $P_{0,0}$, $P_{1,0}$, $P_{2,0}$ and $P_{3,0}$

$$P_{0,3} = (1-t)^3 P_{0,0} + 3t(1-t)^2 P_{1,0} + 3t^2(1-t)P_{2,0} + t^3 P_{3,0}.$$



degree of Bézier curve

The general form of a degree n Bézier curve defined by the control points P_i (where $i = 0, 1, \ldots, n$) is

$$C(t) = \sum_{i=0}^{n} b_{i,n}(t) P_i,$$

where $b_{i,n}(t)$ are called **Bernstein polynomials** that are defined using

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

and $\binom{n}{i}$ is the Binomial coefficient.

The Binomial Coefficient

The **Binomial coefficient** is written using $\binom{n}{i}$ and is read as "n choose i" since it gives the number of ways of choosing i items from a set of n items

$$\binom{n}{i} = \frac{n!}{i!(n-i)!},$$

where $n!$ denotes the factorial of n

Pascal's triangle

$n = 0$:					1				
$n = 1$:				1		1			
$n = 2$:			1		2		1		
$n = 3$:		1		3		3		1	
$n = 4$:	1		4		6		4		1

Case of cubic Bernstein Polynomials

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

For example, the Bernstein polynomials for a cubic Bézier curve are

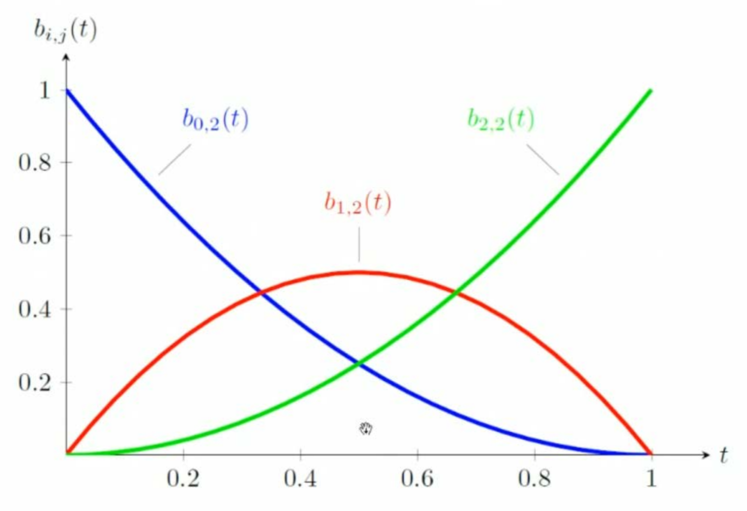
$$b_{0,3}(t) = \binom{3}{0} t^0 (1-t)^{3-0} = (1-t)^3,$$

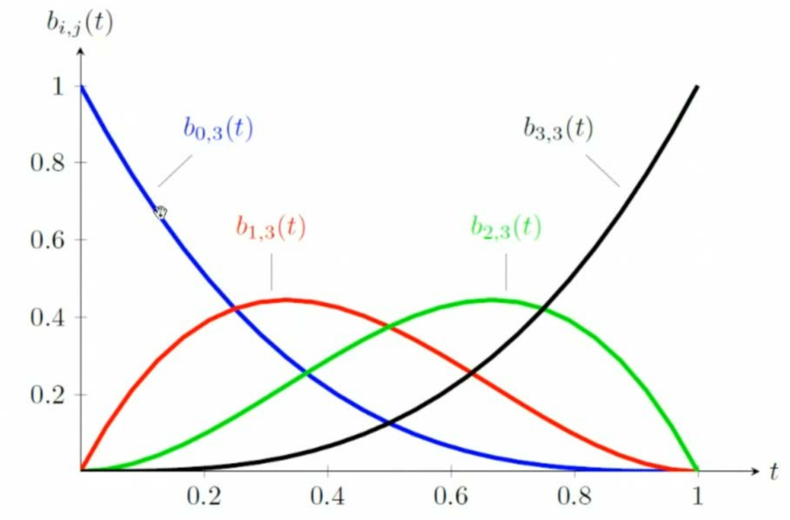
$$b_{1,3}(t) = \binom{3}{1} t^1 (1-t)^{3-1} = 3t(1-t)^2,$$

$$b_{2,3}(t) = \binom{3}{2} t^2 (1-t)^{3-2} = 3t^2(1-t),$$

$$b_{3,3}(t) = \binom{3}{3} t^3 (1-t)^{3-3} = t^3,$$

Behavior of Bernstein Polynomials





Bézier curve in Matrix form

In order to save computational effort, Bézier curves are precalculated and expressed in matrix form as follows:

$$C(t) = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-1} & P_n \end{pmatrix} M \begin{pmatrix} t^n \\ t^{n-1} \\ \vdots \\ t \\ 1 \end{pmatrix},$$

where M is an $(n+1) \times (n+1)$ matrix.

Consider the quadratic Bézier curve with the brackets expanded out

$$C(t) = (t^2 - 2t + 1)P_0 + (-2t^2 + 2t)P_1 + t^2 P_2,$$

this can be expressed in matrix form as

$$C(t) = \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}.$$

Similarly a cubic Bézier curve can be expressed using

$$C(t) = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}.$$

Bézier curve Properties

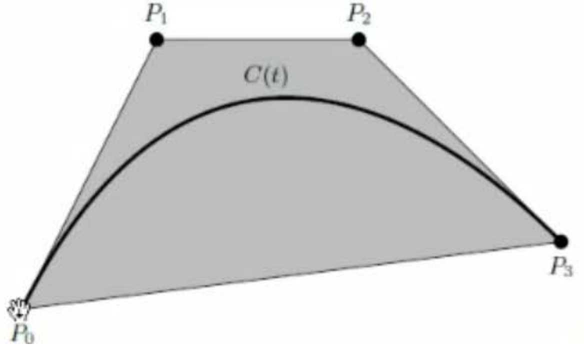
a Bézier curve begins at point P_0 and ends at point P_n;

a Bézier curve is a straight line if and only if it is possible to draw a straight line through all of the control points;

the start and end of a Bézier curve is tangental to the start and end section of the control polygon;

a Bézier curve can be split into two Bézier curves;

a Bézier curve is contained within its control polygon. This is known as the **convex hull property**



Examples with changing control points