Computer Graphics

Lecture 06

Inverse Transformations

To invert a scaling matrix, scale by the reciprocals of the scale factors, assuming they are all nonzero.

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}^{-1} = \begin{bmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{bmatrix}.$$

To invert a rotation through angle θ, rotate through angle $-\theta$ (about the same axis in the case of $\mathbf{R}^3$). Equivalently, transpose the matrix:

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R_\theta^T$$

A reflection is its own inverse.

Composite Transformations and Homogeneous Coordinates

The composition of two linear transformations is represented by the product of the corresponding matrices. The OpenGL modelview transformation is usually a composition of several affine transformations. It is more efficient to compute and store the composition as a single matrix and apply it to the vertices than to apply each operator individually to all the vertices.

This raises the question of how to represent a translation operator as a matrix.
The solution lies in the use of homogeneous coordinates, which also allow projective transformations to be represented by matrices.

Homogeneous Coordinates

The point with Cartesian coordinates (x; y; z) has homogeneous coordinates (x; y; z; 1), or more generally,

α*(x; y; z; 1) for any nonzero scalar α

This equivalence class of all points in R^4 that project to (x; y; z; 1) (with center of projection at the origin and projection plane w = 1).

Reversing the mapping, the Cartesian coordinates of the point with homogeneous coordinates (x; y; z;w) are (x/w; y/w; z/w) obtained by scaling by 1/w and dropping the fourth component (1).

Affine Transformations in Homogeneous Coordinates

For the purpose of representing affine transformations, we always have $w = 1$. Then the matrix representation of T for $T\mathbf{p} = A\mathbf{p} + \mathbf{t}$ is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The blocks of the matrix are as follows.

- Upper left 3 by 3 block: linear operator A
- Upper right 3 by 1 block: translation vector $\mathbf{t}$
- Lower left 1 by 3 block: zero row vector $\mathbf{0}^T$
- Lower right 1 by 1 block: scalar with value 1

For a pure translation, we take $A = I$ so that $T\mathbf{p} = \mathbf{p} + \mathbf{t}$, and for a linear operator, we have $\mathbf{t} = \mathbf{0}$ so that $T\mathbf{p} = A\mathbf{p}$.

Inverse Transformations in Homogeneous Coordinates

Note that T maps a point $\mathbf{p}$ in homogeneous coordinates to the transformed point in homogeneous coordinates.

$$\begin{bmatrix} T\mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{p} + \mathbf{t} \\ 1 \end{bmatrix}.$$

The sequence of operations represented by T is (1) apply the linear operator A, and then (2) translate by $\mathbf{t}$. We invert this by first translating by $-\mathbf{t}$, and then applying A^{-1} to obtain $A^{-1}(\mathbf{p} - \mathbf{t})$ when applied to $\mathbf{p}$. This is equivalent to applying A^{-1} and then translating by $-A^{-1}\mathbf{t}$. Hence the inverse operator is

$$\begin{bmatrix} A^{-1} & -A^{-1}\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

We can verify this as follows.

$$\begin{bmatrix} A^{-1} & -A^{-1}\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Composite Transformation Examples

Problem Construct the transformation that rotates a square in the x-y plane through angle θ about its center $\mathbf{c} = (c_x, c_y)$.

Solution We will simplify the notation by omitting z components; i.e., we use homogeneous coordinates for points in $\mathbf{R}^2$.

1. Translate by $-\mathbf{c}$: $T_{-\mathbf{c}} = \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$
2. Rotate through angle θ: $R_\theta = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where

   $C = \cos\theta$ and $S = \sin\theta$
3. Translate by $\mathbf{c}$: $T_{\mathbf{c}} = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix}$

Composite Transformation Examples

The composite transformation is

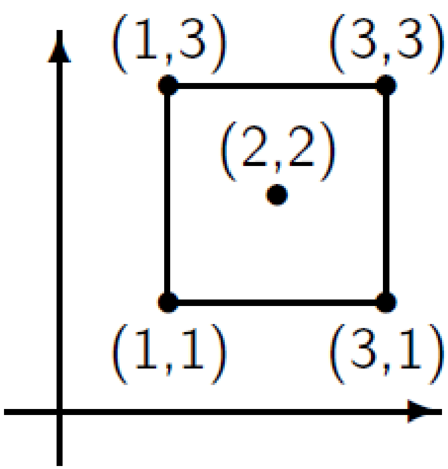
$$
\begin{aligned}
T = T_{\mathbf{c}} R_\theta T_{-\mathbf{c}} &= \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C & -S & c_x \\ S & C & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C & -S & -Cc_x + Sc_y + c_x \\ S & C & -Sc_x - Cc_y + c_y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow
\end{aligned}
$$

$$T(\mathbf{p}) = R(\mathbf{p}) - R(\mathbf{c}) + c = R(\mathbf{p} - \mathbf{c}) + \mathbf{c}.$$

Composite Transformation Examples

Let $\theta = \pi/2$ so that $C = 0$ and $S = 1$, and let $c_x = c_y = 2$. Then

$$T = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix},$$



$$T\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Composite Transformation Examples

Problem Consider a unit square (side length 1) centered at $\mathbf{c} = (10, 10)$ with sides parallel to the axes. Construct a matrix T that rotates the square through $\theta = 20$ degrees clockwise about its center, and scales the side lengths uniformly by α without changing the location of the center. Let $C = \cos(20^\circ)$, $S = \sin(20^\circ)$.

Solution $T = T_{-\mathbf{c}}^{-1} S_\alpha R_\theta^{-1} T_{-\mathbf{c}} =$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \alpha C & \alpha S & 10(1 - \alpha C - \alpha S) \\ -\alpha S & \alpha C & 10(1 - \alpha C + \alpha S) \\ 0 & 0 & 1 \end{bmatrix}.$$

Composite Transformation Examples

Problem Describe the effect of applying 36 successive transformations by T to the square by specifying the resulting shape, size, orientation, and location.

Solution

$$
\begin{aligned}
T^k &= (T_{-\mathbf{c}}^{-1} S_\alpha R_\theta^{-1} T_{-\mathbf{c}})^k = T_{-\mathbf{c}}^{-1} (S_\alpha R_\theta^T)^k T_{-\mathbf{c}} \\
&= T_{-\mathbf{c}}^{-1} S_\alpha^k R_{-\theta}^k T_{-\mathbf{c}} \text{ since } S_\alpha R_\theta^T = R_\theta^T S_\alpha \\
&= T_{-\mathbf{c}}^{-1} S_{\alpha^k} R_{-k\theta} T_{-\mathbf{c}}
\end{aligned}
$$

Thus, T scales the sides by α^k and rotates the square through angle $k\theta = 720^\circ$ clockwise about the center. The shape is square; side lengths are α^{36}; the sides remain parallel to the axes, and the center remains at (10,10). Note that the scaling and rotation operators commute because the scaling is uniform ($S_\alpha = \alpha I$).

Composite Rotation

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{bmatrix} \quad R_y = \begin{bmatrix} C_1 & 0 & S_1 \\ 0 & 1 & 0 \\ -S_1 & 0 & C_1 \end{bmatrix}, \quad R_z = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Rotation matrix = ?

http://what-when-how.com/wp-content/uploads/2012/07/

Homogeneous Composite Rotation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi\cos\theta & \sin\psi\sin\varphi\cos\theta - \cos\psi\sin\theta & \cos\psi\sin\varphi\cos\theta + \sin\psi\sin\theta & 0 \\ \cos\varphi\sin\theta & \sin\psi\sin\varphi\sin\theta + \cos\psi\cos\theta & \cos\psi\sin\varphi\sin\theta - \sin\psi\cos\theta & 0 \\ -\sin\varphi & \sin\psi\cos\varphi & \cos\psi\cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$