Computer Graphics

Lecture 05

Cofactor Matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If A is a square matrix then cofactor matrix C is given by

$$\mathrm{C} = \begin{pmatrix} (-1)^{i+j}\,d & (-1)^{i+j}\,c \\ (-1)^{i+j}\,b & (-1)^{i+j}\,a \end{pmatrix}$$

Where *i* and *j* are corresponding row and column

$$= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Cofactor Matrix

Consider a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Its cofactor matrix is

$$\mathbf{C} = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ & & \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ & & \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

where $\begin{vmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{vmatrix} = \det \begin{pmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{pmatrix}$.

Adjoints (Adjugate of a Matrix)

The adjoint is defined as it is so that the product of matrix A with its adjoint yields a diagonal matrix whose diagonal entries are det(A).

Aadj(A) = det(A)I

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ The adjoint of the 2 × 2 matrix $\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

It is seen that det(adj(A)) = det(A) and hence that adj(adj(A)) = A

Adjoints

Consider a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Its adjoint is given by

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}} = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ & & \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ & & \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

Where C is the cofactor Matrix of A and adj(A) is transpose of the cofactor matrix

Inverse of a Matrix

The inverse of a matrix, M, denoted M^{-1}, which is dealt with here, exists only for square matrices with $\det(M) \neq 0$.

If all elements of the matrix under consideration are real scalars, then it suffices to show that $MN = \mathrm{I}$ and $NM = \mathrm{I}$, where then $N = M^{-1}$.

$$\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1} \,,$$
$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\,\mathrm{adj}(\mathbf{A}) \,.$$

Inverse of a Matrix

$$\mathsf{A} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the matrix inverse is

$$\begin{aligned} \mathsf{A}^{-1} &= \frac{1}{|\mathsf{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{a\,d - b\,c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \end{aligned}$$

Transformation Algebra

Linear Transformations

A point with Cartesian coordinates (x; y; z) is taken to be a column vector (3 by 1 matrix) $p = (x\ y\ z)^T$ for purposes of applying transformations. A linear transformation L on R^3 is represented by a 3 by 3 matrix A. In fact, there is a 1-1 correspondence between linear transformations and matrices, so that L and A are often used interchangeably. The transformed point $p' = L(p) = Ap$ is computed as a matrix-vector product

$$p' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Ap = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It is easily shown that $L(\alpha p) = \alpha\, L(p)$ and $L(p+q) = L(p)+L(q)$, thus defining L as linear. Conversely, to construct the matrix A associated with some linear operator L, the trick is to take the columns of A to be $L(e_j)$, where e_j's are the standard basis vectors.

Affine Transformation

Defn An affine transformation is a linear transformation followed by a translation --

A transformation of the form $T(p) = Tp = Ap + t$, where A is a matrix, and t is a translation vector. In the case of a transformation T from R to R, T is a linear polynomial (but not a linear transformation unless $t = 0$).

An important property of an affine transformation, crucial for computer graphics, is that it maps lines to lines and hence planes to planes. This means that we can transform a line segment or triangle by transforming its vertices.

Use of Affine Transformations

The fact that all transformations in the OpenGL vertex pipeline map triangles to triangles is what allows fragment depths and color intensities to be computed by linear interpolation from vertex values.

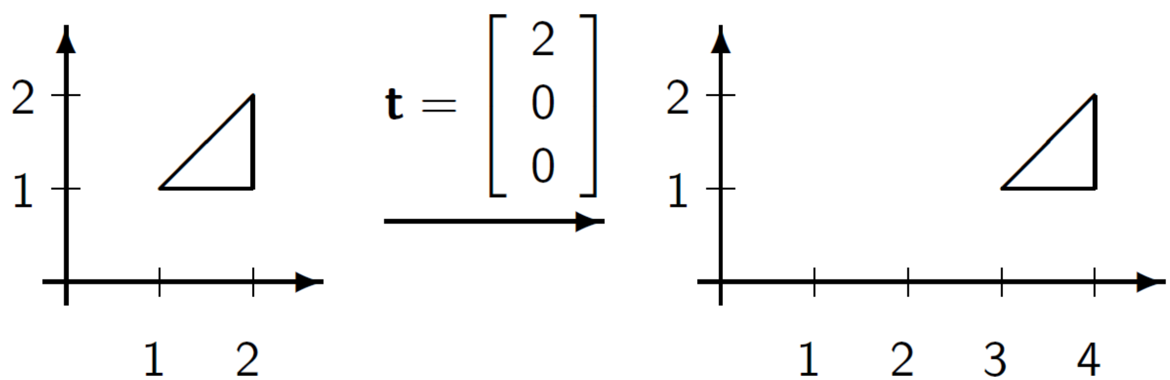
The affine transformations of primary interest are translations and the following four linear operators.

- Scaling
- Rotation
- Reflection
- Orthogonal Projection

Translation

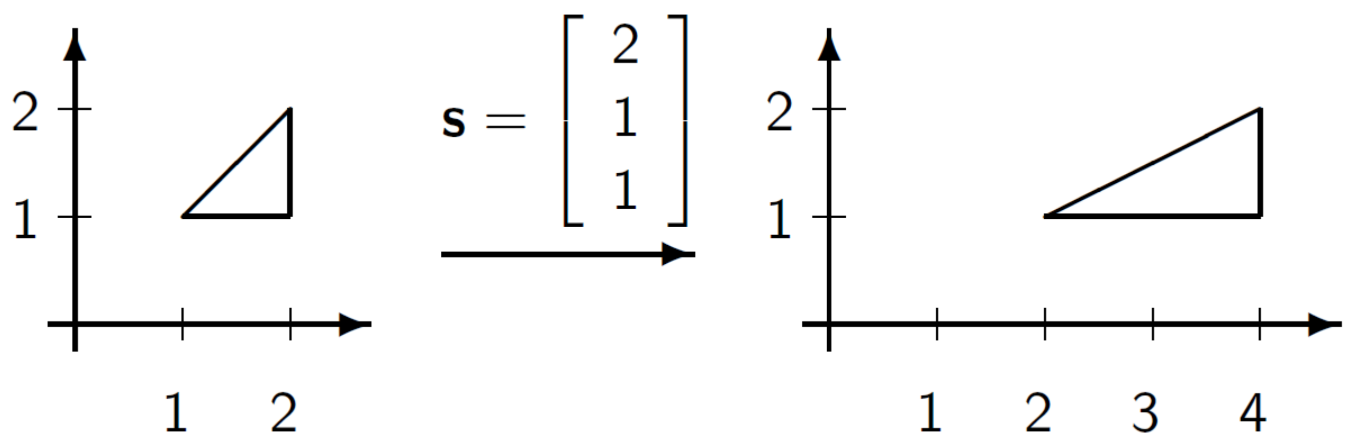
A translation is an affine transformation in which the linear part is the identity matrix I :

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = I\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \end{bmatrix}$$



Scaling

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \end{bmatrix}$$



Note that scaling an object changes not only its size, but also its distance from the origin.

Scaling Relative to a Point

Defn A uniform scaling operator is a scalar times the identity matrix. It scales all components by the same factor.

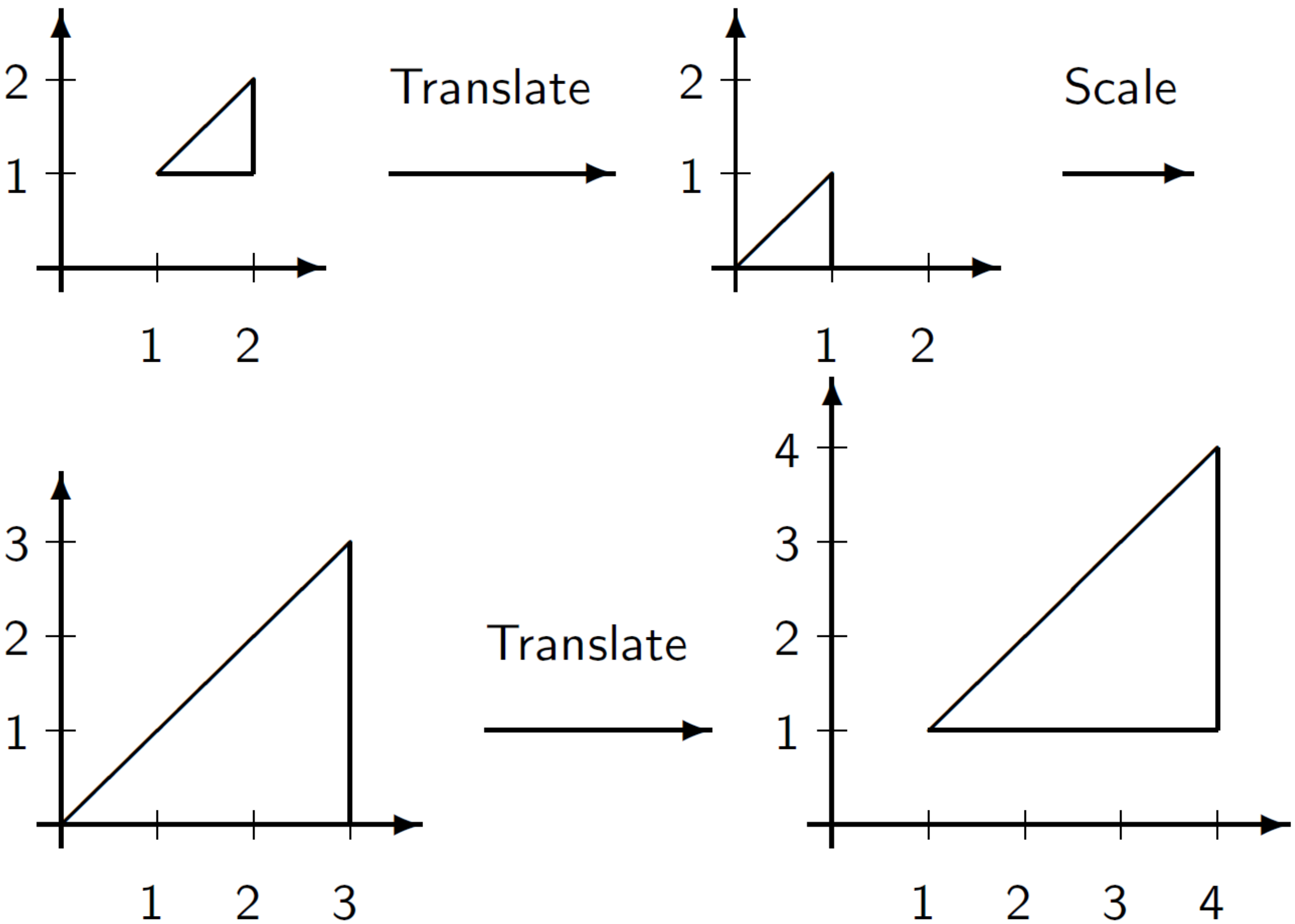
Problem Uniformly scale the size of a triangle by 3 without moving vertex $p_1 = (1; 1; 0)$.

Solution

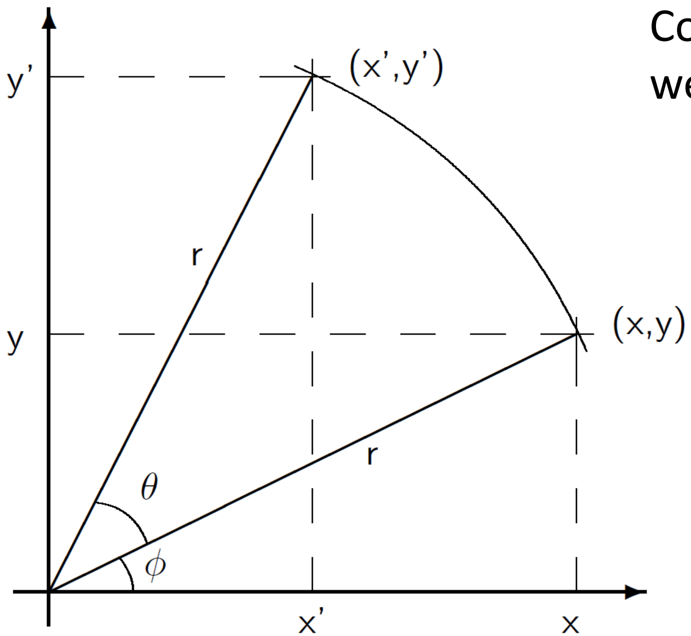
A. Translate by $-p_1$
B. Scale by $3I$
C. Invert the translation: translate by p_1

Note that p_1 is not altered by the sequence of operations:
$p1 \rightarrow p1 - p1 = 0 \rightarrow S0 = 0 \rightarrow 0 + p1 = p1$

Scaling Relative to a Point Example



Planar Rotation: Geometry



Converting polar coordinates to Cartesian coordinates, we have

$$x = r\cos(\phi), \quad y = r\sin(\phi)$$

$$x' = r\cos(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi) = \cos\theta x - \sin\theta y$$

$$y' = r\sin(\theta+\phi) = r(\cos\theta\sin\phi + \sin\theta\cos\phi) = \cos\theta y + \sin\theta x$$

Thus

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for $C = \cos\theta$ and $S = \sin\theta$.

Rotations and Orthogonal Matrices

Defn An orthogonal matrix is a square matrix R with orthonormal columns: That is $R^T R = I$

Defn A rotation is an orthogonal matrix with determinant 1. A planar rotation is a matrix of the form

$$R = \begin{bmatrix} C & -S \\ S & C \end{bmatrix},$$

where $C^2 + S^2 = 1$. R rotates CCW through angle $\theta = \cos^{-1}(C)$ $= \sin^{-1}(S)$.

To see that R is orthogonal, note that

$$R^T R = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} = \begin{bmatrix} C^2 + S^2 & -CS + SC \\ -SC + CS & S^2 + C^2 \end{bmatrix} = I$$

Planar Rotation about a Point

Problem Construct the affine transformation that rotates CCW through angle θ about a point p in the plane.

Solution

a. Translate by -p
b. Rotate through angle θ about the origin
c. Translate by p

The fixed point of the operator is p:

$$\mathbf{p} \longrightarrow \mathbf{p} - \mathbf{p} = \mathbf{0} \longrightarrow R\mathbf{0} = \mathbf{0} \longrightarrow \mathbf{0} + \mathbf{p} = \mathbf{p}$$

Rotations in R^3

There are simple formulas for the axis rotations. To rotate about the x axis, we leave the x-component unaltered, and apply a planar rotation to the (y,z) pair:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{bmatrix} \Rightarrow R_x \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ Cy - Sz \\ Sy + Cz \end{bmatrix},$$

where $C^2 + S^2 = 1$. Similarly,

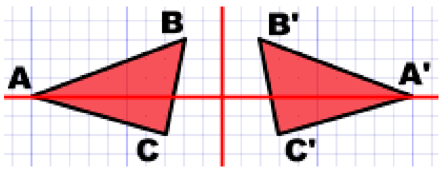
$$R_y = \begin{bmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{bmatrix}, \quad R_z = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

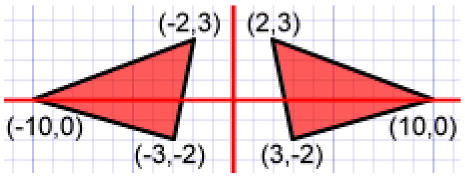
Note that the patterns are consistent with the cyclical ordering of components, corresponding to pairs (y,z), (z,x), and (x,y).

Reflection

Defn A reflection R is a symmetric orthogonal matrix, and hence involuntary (equal to its inverse): $R^2 = I$. As an operator on R^n, it reflects about an (n-1)-dimensional subspace (hyperplane).

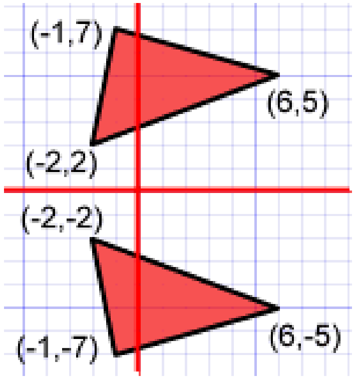
Here the original is **ABC** and the reflected image is **A'B'C'**





Y-Axis

When the mirror line is the y-axis we change each (x,y) into (−x,y)



X-Axis

When the mirror line is the x-axis we change each (x,y) into (x,−y)

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Note that reflecting twice restores the original point ($R^2 = I$)