Computer Graphics

Lecture 04

Certain Special Vectors

In $\mathbf{R}^2$, any vector $[a \quad b]^{\mathbf{T}}$ can be expressed in the form

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

In general, in R^n, the name e_i refers to a vector whose entries are zeroes, except the i^{th}, which is 1

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{and} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Length of a Vector

The length (or **norm**) of the vector $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is the square root of the sum of the squares of the entries of $\mathbf{v}$. If $\mathbf{v} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^{\mathbf{T}}$, then $\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. This corresponds, when we think of $\mathbf{v}$ as a displacement, to the distance that we moved. A vector whose length is 1 is called a **unit vector.**

You can convert a nonzero vector $\mathbf{v}$ to a unit vector, which is called **normalizing** it, by dividing it by its length. We write

$$\mathcal{S}(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$$

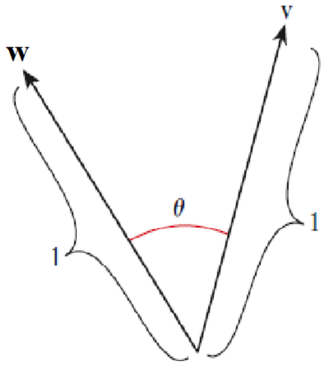
Dot Product

dot product of two n-vectors v and w, defined by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$

This is sometimes denoted <v,w>; in this form, it's usually called the inner product. The dot product is used for measuring angles. If v and w are unit vectors, then

$$\mathbf{v} \cdot \mathbf{w} = \cos(\theta),$$ where θ is the angle between the vectors



This is most often used in the form

$$\theta = \cos^{-1} \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

Dot Product

For a Euclidean space we may also compute the *dot product* of two vectors **u** and **v**. The dot product is denoted **u** · **v**, and its definition is shown below:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=0}^{n-1} u_i v_i \qquad \textbf{(dot product)}$$

For the dot product we have the rules:

(i) $\mathbf{u} \cdot \mathbf{u} \geq 0$, with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = (0, 0, \ldots, 0) = \mathbf{0}$

(ii) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(iii) $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v})$

(iv) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **(commutativity)**

(v) $\mathbf{u} \cdot \mathbf{v} = 0 \Longleftrightarrow \mathbf{u} \perp \mathbf{v}$.

Dot Product

if the dot product is zero then the vectors are orthogonal (perpendicular).

The norm of a vector, denoted ||**u**||, is a nonnegative number that can be expressed using the dot product:

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\left(\sum_{i=0}^{n-1} u_i^2\right)} \quad \textbf{(norm)}$$

Cross Product

The cross product is usually defined for pairs of vectors in 3-space as follows:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \times \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}$$

The cross product is anticommutative that is, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

Cross Product

One of the main uses of the cross product is that

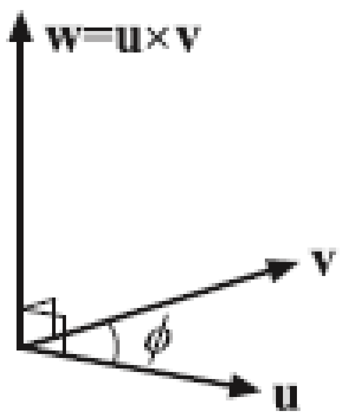
$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\||\sin\theta|$$

where θ is the angle between v and w.

That means that half the length of the cross product is the area of the triangle with vertices (0, 0, 0), (v_x, v_y, v_z), and (w_x,w_y,w_z).

Cross Product

$||\mathbf{w}|| = ||\mathbf{u}\times\mathbf{v}|| = ||\mathbf{u}||\ ||\mathbf{v}|| \sin \phi$, where φ is the smallest angle between **u** and **v**



$\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$.

u, v, w form a right-handed system

u × v = 0 if and only if u || v
(i.e., u and v are parallel), since then sin ϕ = 0

Cross Product

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \qquad \text{(anti-commutativity)}$$

$$(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = a(\mathbf{u} \times \mathbf{w}) + b(\mathbf{v} \times \mathbf{w}) \qquad \text{(linearity)}$$

$$\left.\begin{aligned} &(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \\ =\ &(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} \\ = -&(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = -(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} \end{aligned}\right\} \qquad \text{(scalar triple product)}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \qquad \text{(vector triple product)}$$

Matrices

A matrix, **M**, can be used as a tool for manipulating vectors and points.

M is described by p × q scalars where m_{ij}, $0 \le i \le p - 1$, $0 \le j \le q - 1$, with p rows and q columns

$$\mathbf{M} = \begin{pmatrix} m_{00} & m_{01} & \cdots & m_{0,q-1} \\ m_{10} & m_{11} & \cdots & m_{1,q-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p-1,0} & m_{p-1,1} & \cdots & m_{p-1,q-1} \end{pmatrix} = [m_{ij}].$$

Identity Matrix

Matrix called the *unit matrix*, $\mathbf{I}$, which is square and contains ones in the diagonal and zeros elsewhere.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

When ***A*** is m×n, it is a property of matrix multiplication that $I_m A = A I_n = A$

Matrix-Matrix Addition

Adding two matrices, say M and N, is possible only for equal-sized matrices and is defined as

$$\mathbf{M} + \mathbf{N} = [m_{ij}] + [n_{ij}] = [m_{ij} + n_{ij}]$$

This is, componentwise addition, very similar to vector-vector addition.

Scalar-Matrix Multiplication

A scalar a and a matrix, $\mathbf{M}$, can be multiplied to form a new matrix of the same size as $\mathbf{M}$, which is

$a\mathbf{M} = [am_{ij}]$

Rules:

i) $0\mathbf{M} = \mathbf{0}$

ii) $1\mathbf{M}=\mathbf{M}$

iii) $a(b\mathbf{M}) = (ab)\mathbf{M}$

iv) $(a+b)\mathbf{M}= a\mathbf{M}+b\mathbf{M}$

v) $a(\mathbf{M}+\mathbf{N}) = a\mathbf{M}+ a\mathbf{N}$

Transpose of a Matrix

M^T is the notation for the transpose of $M = [m_{ij}]$, and the definition is $M^T = [m_{ji}]$

The columns become rows and the rows become columns.

Rules:

i) $(a\mathbf{M})^T = a\mathbf{M}^T$

ii) $(\mathbf{M} + \mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$

iii) $(\mathbf{M}^T)^T = \mathbf{M}$

iv) $(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T$

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Trace of a Matrix

The trace of a matrix, denoted tr(M), is simply the sum of the diagonal elements of a square matrix

$$\mathrm{tr}(\mathrm{M}) = \sum_{i=0}^{n-1} m_{ii}.$$

For square matrices A and B, it is true that

$\mathrm{Tr}(A) = \mathrm{Tr}(A^T)$
$\mathrm{Tr}(A+B) = \mathrm{Tr}(A)+\mathrm{Tr}(B)$
$\mathrm{Tr}(aA) = a\mathrm{Tr}(A)$

Matrix Multiplication

This operation, denoted MN between M and N, is defined only if M is of size p × q and N is of size q × r, in which case the result, T, becomes a p × r sized matrix. Mathematically, for these matrices the operation is as follows:

$$\mathbf{T} = \mathbf{MN} = \begin{pmatrix} m_{00} & \cdots & m_{0,q-1} \\ \vdots & \ddots & \vdots \\ m_{p-1,0} & \cdots & m_{p-1,q-1} \end{pmatrix} \begin{pmatrix} n_{00} & \cdots & n_{0,r-1} \\ \vdots & \ddots & \vdots \\ n_{q-1,0} & \cdots & n_{q-1,r-1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{q-1} m_{0,i} n_{i,0} & \cdots & \sum_{i=0}^{q-1} m_{0,i} n_{i,r-1} \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{q-1} m_{p-1,i} n_{i,0} & \cdots & \sum_{i=0}^{q-1} m_{p-1,i} n_{i,r-1} \end{pmatrix}.$$

In other words, each row of M and column of N are combined using a dot product, and the result placed in the corresponding row and column element.

Matrix Multiplication

Rules:

i) $(LM)N = L(MN)$,

ii) $(L+M)N = LN+MN$,

iii) $MI = IM = M$

If the dimensions of the matrices are the same, then $MN \neq NM$

Determinant of a Matrix

The determinant is defined only for square matrices. The determinant of **M**, written |**M**|

$$|\mathrm{M}| = \begin{vmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{vmatrix} = m_{00}m_{11} - m_{01}m_{10}$$

$$|\mathrm{M}| = \begin{vmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{vmatrix}$$

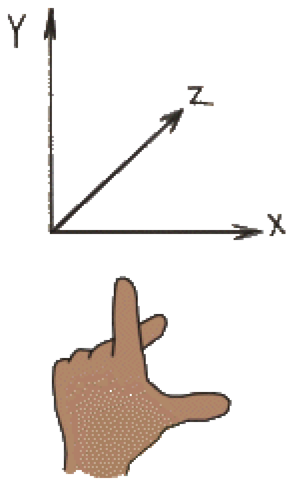
$$= m_{00}m_{11}m_{22} + m_{01}m_{12}m_{20} + m_{02}m_{10}m_{21} - m_{02}m_{11}m_{20} - m_{01}m_{10}m_{22} - m_{00}m_{12}m_{21}$$

If $n \times n$ is the size of M, then the following apply to determinant calculations:

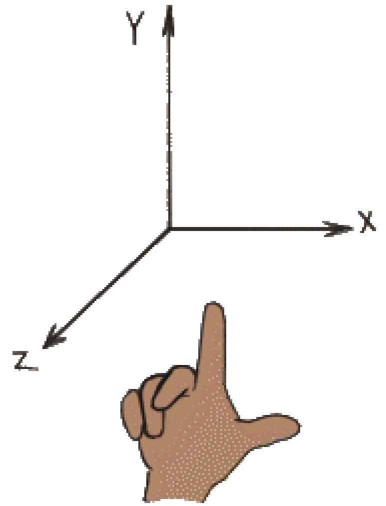
i) $|M^{-1}| = 1/|M|$
ii) $|MN| = |M|\ |N|$
iii) $|aM| = a^n|M|$,
iv) $|MT\ | = |M|$

Determinant of a Matrix

The orientation of a basis can be determined via determinants. A basis is said to form a right-handed system, also called a positively oriented basis, if its determinant is positive. The standard basis has this property, since $|\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_z| = (\mathbf{e}_x \times \mathbf{e}_y) \cdot \mathbf{e}_z = (0,0,1) \cdot \mathbf{e}_z = \mathbf{e}_z \cdot \mathbf{e}_z = 1 > 0$. If the determinant is negative, the basis is called negatively oriented or is said to be forming a left-handed system.



Left hand



Right hand

Thumb indicates the positive direction of x
index indicates the positive direction of y

middle finger pointing out indicates the positive direction of z